

DOUBLE-GENERIC INITIAL IDEAL AND HILBERT SCHEME

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ABSTRACT. Following the approach in the book “Commutative Algebra”, by D. Eisenbud, where the author describes the generic initial ideal by means of a suitable total order on the terms of an exterior power, we introduce first the *generic initial extensor* of a subset of a Grassmannian and then the *double-generic initial ideal* of a so-called *GL-stable subset* of a Hilbert scheme. We discuss the features of these new notions and introduce also a partial order which gives another useful description of them. The double-generic initial ideals turn out to be the appropriate points to understand some geometric properties of a Hilbert scheme: they provide a necessary condition for a Borel ideal to correspond to a point of a given irreducible component, lower bounds for the number of irreducible components in a Hilbert scheme and the maximal Hilbert function in every irreducible component. Moreover, we prove that every isolated component having a smooth double-generic initial ideal is rational. As a byproduct, we prove that the Cohen-Macaulay locus of the Hilbert scheme parameterizing subschemes of codimension 2 is the union of open subsets isomorphic to affine spaces. This improves results by J. Fogarty (1968) and R. Treger (1989).

INTRODUCTION

Let $\mathbf{Hilb}_{p(t)}^n$ denote the Hilbert scheme parameterizing all subschemes in a projective space \mathbb{P}_K^n with Hilbert polynomial $p(t)$ over an infinite field K . The Hilbert scheme was first introduced by Grothendieck [26] in the 60s and, although it has been intensively studied by several authors, its structure and features are still quite mysterious. Also the topological structure of a Hilbert scheme is not well-understood yet. For instance, in general it is not known how many irreducible components a Hilbert scheme has and which of them are rational. In these topics, some few special cases have been treated, for example, by J. Fogarty [24], R. Piene and M. Schlessinger [35], R. Treger [40], P. Lella and M. Roggero [31].

The aim of this paper is to develop algebraic constructive methods in the context of the computation of initial and generic initial ideals, in order to study properties of the Hilbert scheme. Therefore, we often identify a point of $\mathbf{Hilb}_{p(t)}^n$ with any ideal in $S := K[x_0, \dots, x_n]$ defining it as a scheme in \mathbb{P}_K^n and we endow S with a term order. By our techniques, we give some lower bounds for the number of irreducible components of $\mathbf{Hilb}_{p(t)}^n$ and determine the maximal Hilbert function in every irreducible component. Moreover, we obtain some interesting results about the rationality of the components

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of $\mathbf{Hilb}_{p(t)}^n$, in particular for the components that contain a point corresponding to an arithmetically Cohen-Macaulay scheme in codimension two.

The notion of generic initial ideal has attracted the attention of many researchers since its first introduction. Indeed, a generic initial ideal $\text{gin}(I)$ of a homogeneous ideal $I \subset S$ contains many information about the ideal I and about the scheme defined by I (e.g. [23]). It is noteworthy that $\text{gin}(I)$ can be obtained from I by a flat deformation corresponding to a rational curve on the Hilbert scheme. Furthermore, the set of generic initial ideals coincides with that of Borel-fixed ideals, which appear as a useful tool in the investigation of the Hilbert scheme, especially for what concerns its components, already in the 60s [28]. Indeed, “every component and intersection of components of a Hilbert scheme contains at least one Borel-fixed ideal” [36]. This property follows essentially by two facts: the generic initial ideals are Borel-fixed and the irreducible components of $\mathbf{Hilb}_{p(t)}^n$ are invariant under the action of the general linear group $\text{GL} := \text{GL}_K(n+1)$ induced by the action on S .

Every component of $\mathbf{Hilb}_{p(t)}^n$ can contain more than one Borel-fixed ideal. Nevertheless, for any given term order in S and for every irreducible component Y of $\mathbf{Hilb}_{p(t)}^n$, we identify a special Borel-fixed ideal \mathbf{G}_Y , which we call *double-generic initial ideal*, and which gives us information about Y . Roughly speaking, \mathbf{G}_Y is the “generic initial ideal of the generic (and the general) point of Y ” (see Definition 4.5). The introduction and the investigation of the notion of double-generic initial ideal have been inspired by the ideas of both Gröbner strata and marked schemes, which support the main results of this paper, although they do not explicitly appear. In fact, some of our examples have been obtained by the available computational methods already developed in the study of marked schemes by several authors.

It is not easy to get a generic initial ideal: a deterministic computation using parameters can be very heavy, while random changes of coordinates gives a non-certain result. The double-generic initial ideal overcomes these difficulties, since it is easy to individuate \mathbf{G}_Y starting from the list of Borel-fixed ideals on Y : it is sufficient to detect the maximum in this list w.r.t. a suitable order on the Borel-fixed ideals of Y (see Theorem 5.4 and Corollary 5.6).

Along the paper, we consider the classical scheme-theoretic embedding of the Hilbert scheme $\mathbf{Hilb}_{p(t)}^n$ in the Grassmannian $\mathbf{Gr}_{S_m}^q$, where S_m is the vector space of the homogeneous polynomials in S of degree m , for m a sufficiently large degree, and $q := \binom{n+m}{n} - p(m)$. More precisely, it is sufficient to take $m \geq r$, where r is the Gotzmann number of the Hilbert polynomial $p(t)$ (for more details see for instance [14]). Since in turn the Grassmannian $\mathbf{Gr}_{S_m}^q$ can be embedded in $\mathbb{P}(\wedge^q S_m)$ via the Plücker embedding, a point V of $\mathbf{Hilb}_{p(t)}^n$ can be identified with a non-zero totally decomposable tensor, i.e. an *extensor* $f_1 \wedge \cdots \wedge f_q$, where $f_1, \dots, f_q \in S_m$ are linearly independent polynomials such that the ideal (f_1, \dots, f_q) defines the projective subscheme corresponding to V .

In the above setting, we follow the approach presented by D. Eisenbud in his book [18] to deal with the generic initial ideal by means of a suitable total order on the terms of $\wedge^q S_m$, depending on the term order \prec on S (see (2.1)). Thus, we associate to every subset W of $\mathbf{Gr}_{S_m}^q$ a suitable set of terms in $\wedge^q S_m$ called Δ -support of W (see Definition 2.2), and then introduce the initial extensor $\mathbf{in}(W)$ of W as the maximum of the Δ -support of W . Further, we also introduce the generic initial extensor $\mathbf{gin}(W)$ of W as the maximum of the Δ -support of the orbit of W under the usual action of GL on $\mathbf{Gr}_{S_m}^q$ (see Definition 2.4). We prove that $\mathbf{in}(W)$ and $\mathbf{gin}(W)$ do not change when W is replaced by its closure

\overline{W} (Proposition 2.9). In particular, if W is closed and irreducible, $\mathbf{in}(W)$ and $\mathbf{gin}(W)$ can be read as the initial extensor and the generic initial extensor of either the generic point of W or the set of closed points of W (Proposition 2.9, Remark 2.10). Moreover, exploiting the analogous property for ideals given in [18, Theorem 15.18], we prove that $\mathbf{gin}(W)$ is fixed by the Borel subgroup of GL , up to multiplication by a non-null element of K (Theorem 2.7 and Corollary 2.11).

In Section 3, we focus our attention on the subsets W of $\mathbf{Gr}_{S_m}^q$ that are closed and stable under the action of GL and prove that $\mathbf{in}(W)$ and $\mathbf{gin}(W)$ coincide and the corresponding point of the Grassmannian belongs to W (Proposition 3.1). If W is closed and irreducible, then there is a dense open subset of W consisting of points having $\mathbf{gin}(W)$ as generic initial extensor (Proposition 3.3).

In Section 4, we concentrate on subsets of the Hilbert scheme. We prove that there is a perfect correspondence between the notions of initial and generic initial extensor of any point V of $\mathbf{Hilb}_{p(t)}^n$ and those of initial and generic initial ideal of any ideal defining V as a subscheme of \mathbb{P}_K^n (Theorem 4.2 and Corollary 4.3). Furthermore, we prove that if $Y \subseteq \mathbf{Hilb}_{p(t)}^n$ is irreducible, closed and invariant under the action of GL , then the ideal associated to $\mathbf{gin}(Y)$ does not depend on the chosen $m \geq r$ for the embedding in a Grassmannian scheme (Corollaries 4.3 and 4.4). This allows us to define the double generic initial ideal of Y (Definition 4.5). We also present some relevant subsets of $\mathbf{Hilb}_{p(t)}^n$ which are irreducible, closed and invariant under the action of GL (Examples 4.7 and 4.8).

In Section 5, we introduce a suitable partial order $\prec\prec$ on the terms of $\wedge^q S_m$ and prove that the initial extensor and the generic initial extensor of a closed irreducible subset $W \subseteq \mathbf{Gr}_{S_m}^q$ are, respectively, the maxima of the Δ -supports of W and of its orbit with respect to this partial order (see Definition 5.3 and Theorem 5.4). Although a partial order might appear less convenient than a total order, this feature of $\prec\prec$ is in fact a crucial point of the paper, which we exploit in order to obtain some relevant applications. We also explore some of the properties of this partial order, in particular when we consider either the degrevlex term order or a constant Hilbert polynomial (Propositions 5.10 and 5.11).

Finally, in Section 6, we present some interesting applications of the previous results. First, we point out a necessary condition for a Borel term to correspond to a point of a given irreducible component of a Hilbert scheme (Proposition 6.1). Then, we obtain lower bounds on the number of irreducible components of $\mathbf{Hilb}_{p(t)}^n$ simply counting the maximal elements with respect to $\prec\prec$ among the extensor terms corresponding to the Borel-fixed ideals in $\mathbf{Hilb}_{p(t)}^n$. We observe that this bound depends on the chosen term order on S (Proposition 6.2 and Example 6.5). The list of *all* the saturated Borel-fixed ideals in $\mathbf{Hilb}_{p(t)}^n$ can be obtained by the algorithms presented in [15, 30] in the characteristic zero case, and in [6] for every characteristic.

Recall that, for every irreducible, closed subset $Y \subseteq \mathbf{Hilb}_{p(t)}^n$ that is stable under the action of GL , there is a maximum among the Hilbert functions of its points (Remark 6.8). We prove that this maximum is reached by the point corresponding to the double-generic initial ideal \mathbf{G}_Y , hence by the maximum with respect to $\prec\prec$, when we choose the degrevlex term order on S (Theorem 6.6). We conjecture that an analogous result holds for minimal Hilbert functions, when the deglex term order is chosen.

We conclude by investigating the rationality of some components of a Hilbert scheme. We prove that, if Y is an isolated irreducible component of $\mathbf{Hilb}_{p(t)}^n$ and $\mathbf{gin}(Y)$ corresponds to a smooth point in Y , then Y is rational (Theorem 6.10). In the final Example 6.12, we exhibit a Hilbert scheme having two components with the same double-generic initial ideal which is smooth on each component, but singular on the Hilbert scheme. Thus, by Theorem 6.10 both these components are rational. As a relevant consequence of Theorem 6.10, by exploiting [21, Théorème 2(i)] we prove that every component of $\mathbf{Hilb}_{p(t)}^n$ containing a Cohen-Macaulay point of codimension 2 is rational (Corollary 6.11). More precisely, the Cohen-Macaulay locus of such a component is the orbit of an open subset isomorphic to an affine space under the action of GL . This improves results of J. Fogarty [24, Theorem 2.4 and Corollary 2.6] and R. Treger [40, Theorem 2.6] by independent arguments. All these results on rationality give a partial answer to one of the open questions on Hilbert schemes collected at the AIM workshop Components of Hilbert Schemes (Palo Alto, July 19 to 23, 2010) [1, Problem 1.45].

1. GENERALITIES

Let $S := K[x_0, \dots, x_n]$ be the polynomial ring over an infinite field K . For every integer m , S_m denotes the homogeneous component of degree m of S ; if $A \subseteq S$, then A_m denotes $A \cap S_m$. Elements and ideals in S will be always assumed to be homogeneous. A term τ of S is a power product $\tau = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. The set of all the terms of S will be denoted by \mathbb{T} .

We denote by \prec a given term order in S and assume that $x_0 \prec x_1 \prec \cdots \prec x_n$. In our setting, if $\tau = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ and $\sigma = x_0^{\beta_0} \cdots x_n^{\beta_n}$ are two terms in S of the same degree, then

- if \prec is the deglex term order, then $\tau \prec \sigma$ if and only if $\alpha_k < \beta_k$, where $k := \max\{i \in \{0, \dots, n\} \mid \alpha_i \neq \beta_i\}$;
- if \prec is the degrevlex term order, then $\tau \prec \sigma$ if and only if $\alpha_k > \beta_k$, where $k := \min\{i \in \{0, \dots, n\} \mid \alpha_i \neq \beta_i\}$.

If J is a monomial ideal in S , B_J denotes the monomial basis of J , i.e. the set of the terms that are minimal generators of J . For any non-zero polynomial $f \in S$, the *support* $\mathrm{Supp}(f)$ of f is the set of the terms that appear in f with a non-zero coefficient.

The maximal term occurring in the support of f with respect to \prec (w.r.t. \prec) is called the *initial term* of f w.r.t. \prec and denoted by $\mathrm{in}_{\prec}(f)$. If I is an ideal in S , the *initial ideal* $\mathrm{in}_{\prec}(I)$ of I w.r.t. \prec is the ideal generated by the initial terms of the polynomials in I . When there is no ambiguity, we will write $\mathrm{in}(f)$ and $\mathrm{in}(I)$ in place of $\mathrm{in}_{\prec}(f)$ and $\mathrm{in}_{\prec}(I)$.

A set $\{f_1, \dots, f_t\}$ of monic polynomials of an ideal I is the *reduced Gröbner basis* of I , w.r.t. \prec , if $\mathrm{in}(I) = (\mathrm{in}(f_1), \dots, \mathrm{in}(f_t))$ and no term in $\mathrm{Supp}(f_i) \setminus \{\mathrm{in}(f_i)\}$ belongs to $\mathrm{in}(I)$. We refer to [34] for an extended treatment of the theory of Gröbner bases and related topics.

We consider the general linear group $\mathrm{GL}_K(n+1)$ (GL , for short) of the invertible matrices of order $n+1$ with entries in K . If $g = (g_{ij})_{i,j \in \{0, \dots, n\}}$ is a matrix in GL , g acts on the polynomials of S in the following way

$$\begin{aligned} g : S &\rightarrow S \\ f(x_0, \dots, x_n) &\mapsto g(f) = f(\sum_{j=0}^n g_{0j}x_j, \dots, \sum_{j=0}^n g_{nj}x_j) \end{aligned}$$

For every subset $V \subseteq S$, we let $g(V) = \{g(f(x_0, \dots, x_n)) \mid f \in V\}$.

If I is an ideal of S , we can consider $\text{in}(g(I))$. It is well-known that there is an open subset $\mathcal{A} \neq \emptyset$ of GL such that, for every $g \in \mathcal{A}$, $\text{in}(g(I))$ is a constant monomial ideal called the *generic initial ideal* of I and denoted by $\text{gin}_{\prec}(I)$ (or $\text{gin}(I)$ when there is no ambiguity). In our setting, $\text{gin}(I)$ is fixed under the action of the Borel subgroup of upper-triangular invertible matrices, hence it is a Borel-fixed ideal (Borel, for short) (see Galligo's Theorem [25] for $\text{char}(K) = 0$ and [5, Proposition 1] for a positive characteristic).

Note that for every degree m :

$$(1.1) \quad \text{in}(I_m) = \text{in}(I)_m \text{ and } \text{gin}(I_m) = \text{gin}(I)_m.$$

The saturation of the ideal $I \subset S$ is the ideal $I^{\text{sat}} := \cup_{k \geq 0} (I : (x_0, \dots, x_n)^k)$ and I is *saturated* if $I = I^{\text{sat}}$. The schemes $\text{Proj}(S/I)$ and $\text{Proj}(S/I')$ are equal as subschemes of \mathbb{P}_K^n if and only if I and I' have the same saturation.

Given a homogeneous ideal I in S , we refer to [19, Chapters 1 and 4] for the definitions of the Hilbert function (denoted by $H_{S/I}$), Hilbert polynomial and Castelnuovo-Mumford regularity, or simply regularity (denoted by $\text{reg}(I)$). Here, we recall that the regularity $\text{reg}(X)$ of the scheme $X = \text{Proj}(S/I)$ is the regularity of the ideal I^{sat} and $\text{reg}(I) \geq \text{reg}(I^{\text{sat}})$. Moreover, for every $m \geq \text{reg}(I)$, we say that I is *m-regular* and the Hilbert function $H_{S/I}$ satisfies $H_{S/I}(m) = p(m)$, where $p(t)$ is its Hilbert polynomial. In this case, we will also say that $p(t)$ is the Hilbert polynomial of I . Recall that, if $m \geq \text{reg}(I)$, then $\text{Proj}(S/I)$ can be completely recovered by the K -vector space I_m , since $(I_m)^{\text{sat}} = I^{\text{sat}}$. If $\text{char}(K) = 0$, the regularity of a Borel-fixed ideal is the maximum of the degrees of its minimal generators. The initial ideal and the generic initial ideal of an ideal I have the same Hilbert function as I . However, their regularities satisfy the inequalities $\text{reg}(\text{in}(I)) \geq \text{reg}(I)$ and $\text{reg}(\text{gin}(I)) \geq \text{reg}(I)$, which sometimes are strict. Moreover, the initial ideal and the generic initial ideal of a saturated ideal can be no more saturated. However, if \prec is the degrevlex term order, then I and $\text{gin}(I)$ share the same regularity and each of them is saturated if the other is (see [4]).

Example 1.1. Let I be the ideal $(x_2^2, x_1x_2 + x_0^2) \subset K[x_0, x_1, x_2]$, with $\text{char}(K) = 0$ (see [33]). The ideal I is saturated and $\text{reg}(I) = 3$, while, for every term order on S , $\text{in}(I) = (x_2^2, x_1x_2, x_0^2x_2, x_0^4)$ is not saturated, and $\text{reg}(\text{in}(I)) = 4$. If \prec is the degrevlex term order, then $\text{gin}(I) = (x_2^2, x_1x_2, x_1^3)$ is a saturated ideal with regularity 3. If \prec is the deglex term order, then $\text{gin}(I) = (x_2^2, x_1x_2, x_0^2x_2, x_0^4)$ is not saturated with $\text{reg}(\text{gin}(I)) = 4$.

2. INITIAL AND GENERIC INITIAL EXTENSORS

Let \prec be a term order on S and m a positive integer. In the present section, we consider S_m as a K -vector space.

The *initial space* of a K -vector space $V \subset S_m$ is the K -vector space $\text{in}(V) := \langle \text{in}(f) | f \in V \rangle$. There is an open subset $\mathcal{A} \neq \emptyset$ of GL such that, for every $g \in \mathcal{A}$, $\text{in}(g(V))$ is constant. This constant initial space is called the *generic initial space* of V and is denoted by $\text{gin}(V)$ (see [23] and the references therein).

Let q be a positive integer and $\mathbf{Gr}_{S_m}^q$ the Grassmannian of subspaces of S_m of dimension q . We consider $\mathbf{Gr}_{S_m}^q$ as a subscheme embedded in the projective space $\mathbb{P}(\wedge^q S_m)$ through the Plücker embedding (for example, see [27]).

Definition 2.1. An *extensor* (of step q) on S_m is a non-zero element of $\wedge^q S_m$ of the form $f_1 \wedge \dots \wedge f_q$, with $f_1, \dots, f_q \in S_m$.

Note that an element $f_1 \wedge \cdots \wedge f_q \in \wedge^q S_m$ vanishes whenever the vector space generated by f_1, \dots, f_q has dimension lower than q .

Following [18, Section 15.9], we say that an *extensor term* (or simply a *term*) in $\wedge^q S_m$ is an extensor of type $\tau_1 \wedge \cdots \wedge \tau_q$, with $\tau_i \in S_m \cap \mathbb{T}$; furthermore, we say that a term of $\wedge^q S_m$ is a *normal expression* if $\tau_1 \succ \cdots \succ \tau_q$. We denote by $\mathbf{T}_{S_m}^q$ the set of the normal expression terms and from now on, whenever we consider a term $\tau_1 \wedge \cdots \wedge \tau_q \in \wedge^q S_m$, we assume that it belongs to $\mathbf{T}_{S_m}^q$. Furthermore, $\mathbf{T}_{S_m}^q$ is the K -vector basis we always consider for the K -vector space $\wedge^q S_m$. We can compare the terms in $\mathbf{T}_{S_m}^q$ lexicographically according to \prec , in the following way

$$(2.1) \quad \tau_1 \wedge \cdots \wedge \tau_q \prec \sigma_1 \wedge \cdots \wedge \sigma_q \Leftrightarrow \exists j \in \{1, \dots, q\} : \tau_i = \sigma_i, \forall i < j, \text{ and } \tau_j \prec \sigma_j.$$

In this setting, for every $L \in \mathbf{T}_{S_m}^q$, there is a unique Plücker coordinate Δ_L on $\mathbf{Gr}_{S_m}^q$ corresponding to L , and vice versa. Moreover, $\mathbf{Gr}_{S_m}^q = \text{Proj}(K[\Delta_L : L \in \mathbf{T}_{S_m}^q]) =: \text{Proj}(K[\Delta])$. Therefore, if $V = \langle f_1, \dots, f_q \rangle \subseteq S_m$ is a K -vector space of dimension q , the extensor $f_1 \wedge \cdots \wedge f_q$ has the unique writing $\sum_{L \in \mathbf{T}_{S_m}^q} c_L L$, with $c_L = \Delta_L(V) \in K$.

For every $L = \tau_1 \wedge \cdots \wedge \tau_q \in \mathbf{T}_{S_m}^q$, we will denote by U_L the standard open set of $\mathbf{Gr}_{S_m}^q$ corresponding to Δ_L , namely the locus of points in $\mathbf{Gr}_{S_m}^q$ where Δ_L is invertible. Moreover, we will denote by $\text{vs}(L)$ the vector space $\langle \tau_1, \dots, \tau_q \rangle$ in $\mathbf{Gr}_{S_m}^q$.

Definition 2.2. If W is a non-empty subset of $\mathbf{Gr}_{S_m}^q$, the Δ -support of W is the following subset of $\mathbf{T}_{S_m}^q$:

$$\Delta\text{Supp}(W) := \{L \in \mathbf{T}_{S_m}^q \mid U_L \cap W \neq \emptyset\}.$$

If $W = \{V\}$, we simply write $\Delta\text{Supp}(V)$ for $\Delta\text{Supp}(\{V\})$.

We will apply this definition and the related ones also to subschemes W of $\mathbf{Gr}_{S_m}^q$, meaning that the Δ -support of a scheme is that of its underlying set of points (see also Example 2.8)

If W is a non-empty subset of $\mathbf{Gr}_{S_m}^q$, then its Δ -support is non-empty. From now, we consider subsets of $\mathbf{Gr}_{S_m}^q$ that are non-empty.

Proposition 2.3. *Let W be a subset of $\mathbf{Gr}_{S_m}^q$.*

- (i) *If \overline{W} is the closure of W , then $\Delta\text{Supp}(W) = \Delta\text{Supp}(\overline{W})$.*
- (ii) *If W is closed and \widetilde{W} is the set of its closed points, then $\Delta\text{Supp}(W) = \Delta\text{Supp}(\widetilde{W})$.*
- (iii) *If W is closed and irreducible and V is its generic point, then $\Delta\text{Supp}(W) = \Delta\text{Supp}(V)$.*

Proof. (i) It is immediate that $\Delta\text{Supp}(W) \subseteq \Delta\text{Supp}(\overline{W})$, because $W \subseteq \overline{W}$. We now prove the other inclusion. If L belongs to $\Delta\text{Supp}(\overline{W})$, then there is at least a point V in $U_L \cap \overline{W}$. Thus, every open neighbourhood of V meets W non-trivially, in particular U_L meets W non-trivially. Hence, L belongs to $\Delta\text{Supp}(W)$.

Items (ii) and (iii) are straightforward consequence of (i), because $\widetilde{\overline{W}} = W$ (for example, see [10, Chapter V, Section 3.4, Theorem 3]) and $\overline{\{V\}} = W$, in the respective hypotheses. \square

The action of GL on S defined in Section 1 induces an action on $\wedge^q S_m$ in the following natural way: if $H = \tau_1 \wedge \cdots \wedge \tau_q$ is a term in $\mathbf{T}_{S_m}^q$, we set $g(H) = g(\tau_1) \wedge \cdots \wedge g(\tau_q) \in \wedge^q S_m$

where $g \in \text{GL}$ and then extend the action to every element in $\wedge^q S_m$ by linearity. Note that in general $g(H)$ does not need to be a term.

In a similar natural way, we obtain an action of GL on $K[\Delta]$ and on $\mathbf{Gr}_{S_m}^q$ (see also [27, Example 10.18]): for every $g \in \text{GL}$ and for every $H \in \mathbf{T}_{S_m}^q$ (hence, every $\Delta_H \in \Delta$), if $g(H) = \sum_{L \in \mathbf{T}_{S_m}^q} c_L L$, then we set $g(\Delta_H) := \sum_{L \in \mathbf{T}_{S_m}^q} c_L \Delta_L$. Thus, for every point V of $\mathbf{Gr}_{S_m}^q$ and for every element g of GL , by $g(V)$ we mean the point of $\text{Proj}(K[\Delta])$ corresponding to the prime ideal $g(\mathfrak{a})$, with \mathfrak{a} the prime ideal defining V . If V is a K -point of $\mathbf{Gr}_{S_m}^q$, i.e. $V = \langle f_1, \dots, f_q \rangle$ with $f_i \in S_m$, then this action on V is exactly the usual action of GL on the polynomials generating V as a K -vector space.

In this context, the orbit of $V \in \mathbf{Gr}_{S_m}^q$ is the set $\mathcal{O}(V) := \{g(V) \mid g \in \text{GL}\}$. If W is a subset of $\mathbf{Gr}_{S_m}^q$, the orbit of W is $\mathcal{O}(W) := \cup_{V \in W} \mathcal{O}(V)$.

Definition 2.4. If W is a subset of $\mathbf{Gr}_{S_m}^q$, the *initial extensor* of W is

$$\mathbf{in}(W) := \max_{\prec} \Delta \text{Supp}(W)$$

and the *generic initial extensor* of W is

$$\mathbf{gin}(W) := \mathbf{in}(\mathcal{O}(W)) = \max_{\prec} \Delta \text{Supp}(\mathcal{O}(W)),$$

where the maximum is taken w.r.t. to the order \prec , as defined in (2.1). If $W = \{V\}$, we simply write $\mathbf{in}(V)$ for $\mathbf{in}(\{V\})$ and $\mathbf{gin}(V)$ for $\mathbf{gin}(\{V\})$.

Remark 2.5. Note that, for every subset W of $\mathbf{Gr}_{S_m}^q$, we have $\mathbf{in}(W) \preceq \mathbf{gin}(W)$ because $\Delta \text{Supp}(W) \subseteq \Delta \text{Supp}(\mathcal{O}(W))$.

Definition 2.6. We say that a term $L \in \mathbf{T}_{S_m}^q$ is *Borel-fixed* (*Borel*, for short) if $g(L) \in \langle L \rangle$ for every upper-triangular matrix $g \in \text{GL}$. We will denote by $\mathbf{B}_{S_m}^q$ the set of Borel terms.

Note that a term $L \in \mathbf{T}_{S_m}^q$ is Borel if and only if the ideal generated by $\mathbf{vs}(L)$ in S is Borel.

Theorem 2.7. Let $V = \langle f_1, \dots, f_q \rangle$ be a K -point of $\mathbf{Gr}_{S_m}^q$ and let $f_1 \wedge \dots \wedge f_q = \sum_{L \in \mathbf{T}_{S_m}^q} c_L L$. Then:

- (i) $\Delta \text{Supp}(V) = \{L \in \mathbf{T}_{S_m}^q \mid c_L \neq 0\}$.
- (ii) $\wedge^q \mathbf{in}(V) = \langle \mathbf{in}(V) \rangle$, or equivalently $\mathbf{in}(V) = \mathbf{vs}(\mathbf{in}(V))$.
- (iii) $\wedge^q \mathbf{gin}(V) = \langle \mathbf{gin}(V) \rangle$, or equivalently $\mathbf{gin}(V) = \mathbf{vs}(\mathbf{gin}(V))$, and $\mathbf{gin}(V)$ is Borel.

Proof. (i) As already observed, for a K -point V and a term L , we have $\Delta_L(V) = c_L \in K$. Hence, V belongs to U_L if and only if $\Delta_L(V)$ is invertible in K , thus if and only if $\Delta_L(V)$ is non-zero.

(ii) If $\mathbf{in}(V) = \langle \tau_1, \dots, \tau_q \rangle$, with $\tau_1 \succ \dots \succ \tau_q$, we can assume that f_1, \dots, f_q are polynomials such that $\mathbf{in}(f_i) = \tau_i$, for every $i \in \{1, \dots, q\}$. Then, exploiting item (i) we see that $\mathbf{in}(V) = \max_{\prec} \Delta \text{Supp}(\langle f_1, \dots, f_q \rangle) = \tau_1 \wedge \dots \wedge \tau_q$.

(iii) We immediately obtain the equality $\wedge^q \mathbf{gin}(V) = \langle \mathbf{gin}(V) \rangle$ and can conclude by [18, Theorem 15.18], taking into account also formula (1.1). \square

By the following example we underline that the definition of Δ -support of a subscheme W of $\mathbf{Gr}_{S_m}^q$ does not depend on the possible non-reduced structure of W , i.e. the Δ -support of W coincides with the Δ -support of W^{red} . Moreover, whereas our definition of

Δ -support can be applied to all the subschemes W of a Grassmannian, the characterization of Δ -support given in Theorem 2.7(i) cannot be extended to every W , even when W can be defined by an extensor.

Example 2.8. Let $S = K[x_0, x_1, x_2]$ be endowed with the degrevlex term order. Let us consider the non-reduced closed subscheme $W \subset \mathbf{Gr}_{S_2}^2 = \text{Proj}(K[\Delta])$ defined by the ideal I that is generated by Δ_L^2 , where $L = x_2^2 \wedge x_1 x_2$, and by all the other Plücker coordinates except Δ_L and $\Delta_{L'}$, where $L' = x_1 x_2 \wedge x_1^2$. Note that W is a non-empty subscheme of the Grassmannian, since the radical of I is not irrelevant. Indeed, W is a double structure on the closed K -point V' with Plücker coordinates all equal to 0, except $\Delta_{L'}$. Then, W does not intersect the standard open subsets of the Grassmannian, except $U_{L'}$. Thus, we obtain $\Delta\text{Supp}(W) = \Delta\text{Supp}(V') = \{L'\}$ and $\mathbf{in}(W) = \mathbf{in}(V') = L'$.

Being a double structure over a closed K -point, W is isomorphic to the scheme $\text{Spec}(K[\varepsilon])$ (where $\varepsilon^2 = 0$). Then, using the functorial language, we can see W as an element of $\underline{\mathbf{Gr}}_{S_2}^2(K[\varepsilon])$, where $\underline{\mathbf{Gr}}_{S_2}^2$ denotes the Grassmann functor (e.g., see [11, formula (2.1) and Section 5]).

Notice that W is a special element of $\underline{\mathbf{Gr}}_{S_2}^2(K[\varepsilon])$, since it is given by the rank 2 free (not only locally free) submodule of $S_2 \otimes_K K[\varepsilon]$ generated by $f_1 = \varepsilon x_2^2 - x_1^2$ and $f_2 = x_1 x_2$. Therefore, we can identify it with the extensor in $\wedge^2(S_2 \otimes_K K[\varepsilon])$

$$f_1 \wedge f_2 = (\varepsilon x_2^2 - x_1^2) \wedge x_1 x_2 = \varepsilon(x_2^2 \wedge x_1 x_2) - (x_1^2 \wedge x_1 x_2) = \varepsilon L + L'.$$

Now we can see that Theorem 2.7(i) cannot be extended to this case; in fact the coefficient of L in $\varepsilon L + L'$ is $\varepsilon \neq 0$, but L does not belong to $\Delta\text{Supp}(W)$.

Proposition 2.9. *Let W be a subset of $\mathbf{Gr}_{S_m}^q$.*

- (i) *If \overline{W} is the closure of W , then $\mathbf{in}(W) = \mathbf{in}(\overline{W})$ and $\mathbf{gin}(W) = \mathbf{gin}(\overline{W})$.*
- (ii) *If W is closed and \widetilde{W} is the set of its closed points, then $\mathbf{in}(W) = \mathbf{in}(\widetilde{W})$ and $\mathbf{gin}(W) = \mathbf{gin}(\widetilde{W})$.*
- (iii) *If W is closed and irreducible and V is its generic point, then $\mathbf{in}(W) = \mathbf{in}(V)$ and $\mathbf{gin}(W) = \mathbf{gin}(V)$.*

Proof. For what concerns the initial extensor, the three statements directly follow from Proposition 2.3. For what concerns the generic initial extensor, the statements in (ii) and (iii) are consequences of (i), since $\overline{\widetilde{W}} = W$ and $\overline{\{V\}} = W$ in the respective hypotheses. Then, it remains to prove the statement about the generic initial extensor in (i). It is sufficient to show that $\Delta\text{Supp}(\mathcal{O}(W)) = \Delta\text{Supp}(\mathcal{O}(\overline{W}))$. We only prove the non-obvious inclusion.

Let L be any term in $\Delta\text{Supp}(\mathcal{O}(\overline{W}))$. By definition, $U_L \cap \mathcal{O}(\overline{W})$ is not empty. More precisely, there are an element $g \in \text{GL}$ and a point $V_1 \in \overline{W}$ such that $g(V_1) \in U_L$. Then, $g(V_1)$ belongs to $U_L \cap \overline{g(W)}$, since $g(\overline{W}) = \overline{g(W)}$. By the definition of closure, this implies $U_L \cap g(W) \neq \emptyset$. Hence, $U_L \cap \mathcal{O}(W) \neq \emptyset$ and $L \in \Delta\text{Supp}(\mathcal{O}(W))$. \square

Remark 2.10.

- (i) In many cases, we will identify a closed subset W of $\mathbf{Gr}_{S_m}^q$ either with the set \widetilde{W} of its closed points or, if W is also irreducible, with its generic point. Indeed, by Proposition 2.9, $\mathbf{in}(W)$ and $\mathbf{gin}(W)$ can be also read as the initial extensor and the generic initial extensor either of \widetilde{W} or, if W is irreducible, of the generic point of W . These facts will be useful because, up to an extension of the base field K to

the residue field K_V (where V is either a suitable point in \widetilde{W} or the generic point of W), they will allow to reduce our arguments to the case of a rational point. Note that, being K infinite, GL is Zariski dense in $\text{GL} \otimes_K K'$ for every extension field K' of K , hence the computation of the generic initial extensor does not change after an extension of the base field. Furthermore, if K is algebraically closed and we can identify W with \widetilde{W} , then we do not need to extend the base field.

- (ii) If W is closed and irreducible, we can read $\mathbf{in}(W)$ and $\mathbf{gin}(W)$ as the initial extensor and the generic initial extensor of a *general point* in W . In fact, consider the two non-empty open subsets $W' := W \cap U_{\mathbf{in}(W)}$ and $W'' := W \cap U_{\mathbf{gin}(W)}$ of W . The sets $\widetilde{W'}$ and $\widetilde{W''}$ of the closed points in W' and W'' , respectively, are both dense in W . By construction, we have $\mathbf{in}(V) = \mathbf{in}(W)$ for every point $V \in W'$ and $\mathbf{gin}(V) = \mathbf{gin}(W)$ for every point $V \in W''$.

Corollary 2.11. *Let W be a subset of $\mathbf{Gr}_{S_m}^q$. Then, $\mathbf{gin}(W)$ belongs to $\mathbf{B}_{S_m}^q$.*

Proof. By Proposition 2.9 we can assume that W is closed. As observed in Remark 2.10, we can replace W by a suitable closed point V of W , which can be considered as a K -point after a possible extension of the base field. We conclude by Theorem 2.7(iii). \square

3. STABLE SUBSETS UNDER THE ACTION OF GL

In this section we focus our attention on the subsets W of $\mathbf{Gr}_{S_m}^q$ that are stable under the action of the group GL , i.e. $g(W) = W$ for every $g \in \text{GL}$. Under this hypothesis on W we see that the initial extensor and the generic initial extensor of W coincide. If, moreover, we assume that W is also closed and irreducible, then we obtain interesting properties of the two open subsets $W \cap \mathcal{V}_{\mathbf{in}(W)}$ and $W \cap \mathcal{U}_{\mathbf{in}(W)}$ of the points of W having $\mathbf{in}(W) = \mathbf{gin}(W)$ as initial extensor and generic initial extensor, respectively (see formula (3.1)).

Let V be a point in $\mathbf{Gr}_{S_m}^q$. The closure $\overline{\mathcal{O}(V)}$ of its orbit $\mathcal{O}(V)$ is irreducible, because $\mathcal{O}(V)$ is irreducible, and is stable under the action of GL because $g(\mathcal{O}(V)) \subseteq \mathcal{O}(V)$ by definition of orbit, for every $g \in \text{GL}$, and hence $g(\overline{\mathcal{O}(V)}) \subseteq \overline{g(\mathcal{O}(V))} \subseteq \overline{\mathcal{O}(V)}$. In particular, every subset W of $\mathbf{Gr}_{S_m}^q$ that is stable under the action of GL is a disjoint union of orbits of its points under the action of GL . If, moreover, W is also closed, it is the union of the closure of these orbits.

Proposition 3.1. *Let $W \subseteq \mathbf{Gr}_{S_m}^q$ be closed and stable under the action of GL . Then:*

- (i) $\mathbf{in}(W) = \mathbf{gin}(W)$.
- (ii) For every $W' \subseteq W$, both $\mathbf{vs}(\mathbf{in}(W'))$ and $\mathbf{vs}(\mathbf{gin}(W'))$ belong to W .
- (iii) If W is reducible, then its irreducible components are stable under the action of GL .
- (iv) If Y_1, \dots, Y_ℓ are irreducible components of W , then every irreducible component of $\cap_{i=1}^\ell Y_i$ is stable under the action of GL .
- (v) Let $U \subseteq W$ be open and stable under the action of GL ; the irreducible components of $W \setminus U$ and those of the closure \overline{U} of U are stable under the action of GL .

Proof. (i) To prove $\mathbf{in}(W) = \mathbf{gin}(W)$ it is enough to observe that in the present hypothesis $W = \mathcal{O}(W)$.

(ii) By Proposition 2.9(i) and (ii), we may assume that W' is closed and choose a closed point $V \in W'$ such that $\mathbf{in}(V) = \mathbf{in}(W')$. Extending the field of scalars, if necessary, we may assume that V is a K -point. Exploiting the term order, we can construct a map

$\varphi: \mathbb{A}_K^1 \rightarrow \mathbf{Gr}_{S_m}^q$ such that $\varphi(1) = V, \varphi(0) = \text{in}(V)$, and for every $c \neq 0, 1$ $\varphi(c) = g_c(V)$ where $g_c \in \text{GL}$ corresponds to a diagonal matrix in which the entries of the diagonal are suitable powers of c (see for instance [3]). Then we conclude, since W is closed and contains the orbits of its points. This same argument applies to $\mathbf{gin}(W')$.

(iii) We have to show that every irreducible component Y of W is stable under the action of GL , namely that $g(Y) = Y$ for every $g \in \text{GL}$. By topological arguments, $g(Y)$ is an irreducible component of W . Let V be a point in Y not belonging to any other irreducible component of W . Then, Y is the only irreducible component of W containing the orbit $\mathcal{O}(V)$. On the other hand, $\mathcal{O}(V) = g(\mathcal{O}(V))$ is contained in $g(Y)$ and, in particular, $V \in g(Y)$. Therefore $Y = g(Y)$.

Item (iv) follows directly from (iii).

We now prove (v). For what concerns $W \setminus U$ it is sufficient to observe that it is closed in $\mathbf{Gr}_{S_m}^q$ and stable under the action of GL . Hence, we can apply (iii). Finally, consider $V \in \overline{U} \setminus U$. The intersection of every open neighbourhood A of V with U is non-empty. Moreover, for every $g \in \text{GL}$, $g(A)$ is an open neighbourhood of $g(V)$ and also its intersection with U is non-empty, because U is stable under the action of GL . Then, \overline{U} is stable under the action of GL too and we again apply (iii). \square

For every $L \in \mathbf{T}_{S_m}^q$, consider the following subsets of $\mathbf{Gr}_{S_m}^q$:

$$(3.1) \quad \mathcal{V}_L := \{V \in \mathbf{Gr}_{S_m}^q \mid \text{in}(V) = L\}, \quad \mathcal{U}_L := \{V \in \mathbf{Gr}_{S_m}^q \mid \mathbf{gin}(V) = L\}.$$

Obviously, $\text{vs}(L)$ belongs to \mathcal{V}_L , and \mathcal{U}_L is non-empty if and only if L belongs to $\mathbf{B}_{S_m}^q$. Thus, from now, when we consider \mathcal{U}_L we assume that L is Borel. It is immediate that \mathcal{U}_L is stable under the action of GL , while in general \mathcal{V}_L is not, even when L is Borel-fixed. We also point out that \mathcal{U}_L does not need to contain $\text{vs}(\text{in}(V))$ for every $V \in \mathcal{U}_L$, even when $\text{in}(V)$ is Borel-fixed, as the following example shows.

Example 3.2. Let us assume $\text{char}(K) = 0$ and consider $\mathbf{Gr}_{S_2}^3$ and the degrevlex term order on $S = K[x_0, x_1, x_2]$ with $x_0 \prec x_1 \prec x_2$. If we take $V = \langle x_2^2, x_0x_2, x_1x_2 + x_1^2 \rangle$, we obtain $L := \mathbf{gin}(V) = x_2^2 \wedge x_1x_2 \wedge x_1^2$ and $L' := \text{in}(V) = x_2^2 \wedge x_1x_2 \wedge x_0x_2$, both elements of $\mathbf{B}_{S_2}^3$. Hence, $V \in \mathcal{U}_L$, but $\text{vs}(\text{in}(V)) \notin \mathcal{U}_L$, because $\mathbf{gin}(\text{vs}(\text{in}(V))) = \text{in}(V)$ being $\text{in}(V)$ Borel-fixed. On the other hand, $V \in \mathcal{V}_{L'}$, but $\text{vs}(\mathbf{gin}(V)) \notin \mathcal{V}_{L'}$.

For every Borel term L we will now examine the relations between the three subsets \mathcal{U}_L , \mathcal{V}_L and \mathcal{U}_L of $\mathbf{Gr}_{S_m}^q$. It is obvious that $\mathcal{V}_L \subseteq \mathcal{U}_L$, by definition of initial extensor (see Definition 2.4), while in general $\mathcal{U}_L \not\subseteq \mathcal{V}_L$. Furthermore, as shown by Example 3.2, we can have both $\mathcal{V}_L \not\subseteq \mathcal{U}_L$ and $\mathcal{U}_L \not\subseteq \mathcal{V}_L$. Some more detailed relations can be obtained taking into account the action of GL .

Proposition 3.3.

- (i) For every $L \in \mathbf{T}_{S_m}^q$, $\mathcal{V}_L = \mathcal{U}_L \setminus \bigcup_{L' \succ L, L' \in \mathbf{T}_{S_m}^q} \mathcal{U}_{L'}$.
- (ii) For every $L \in \mathbf{B}_{S_m}^q$, $\mathcal{U}_L = \mathcal{O}(\mathcal{U}_L) \setminus \bigcup_{L' \succ L, L' \in \mathbf{B}_{S_m}^q} \mathcal{O}(\mathcal{U}_{L'})$.
- (iii) $\{\mathcal{V}_L\}_{L \in \mathbf{T}_{S_m}^q}$ and $\{\mathcal{U}_L\}_{L \in \mathbf{B}_{S_m}^q}$ are two stratifications of $\mathbf{Gr}_{S_m}^q$ consisting of locally closed subsets.

Moreover, if W is a closed and irreducible subset of $\mathbf{Gr}_{S_m}^q$ then

- (a) $W \cap \mathcal{V}_{\text{in}(W)}$ is a dense open subset of W , while $W \cap \mathcal{V}_L$ is empty if $L \succ \text{in}(W)$.
- (b) If W is also stable under the action of GL , then $W \cap \mathcal{U}_{\mathbf{gin}(W)} = W \cap \mathcal{O}(\mathcal{V}_{\mathbf{gin}(W)}) = W \cap \mathcal{O}(\mathcal{V}_{\text{in}(W)})$ is a dense open subset of W , while $W \cap \mathcal{U}_L$ is empty if $L \succ \mathbf{gin}(W)$.

Proof. (i) and (ii) directly follow by the definition of initial extensor and generic initial extensor (see Definition 2.4) and by Corollary 2.11.

For (iii) we observe that the two families $\{\mathcal{V}_L\}_{L \in \mathbf{T}_{S_m}^q}$ and $\{\mathcal{U}_L\}_{L \in \mathbf{B}_{S_m}^q}$ are partitions of the Grassmannian, since every point V of $\mathbf{Gr}_{S_m}^q$ is contained in exactly one set \mathcal{V}_L , the one with $L = \mathbf{in}(V)$, and in exactly one set \mathcal{U}_L , the one with $L = \mathbf{gin}(V)$. Moreover, by the previous items it follows that \mathcal{V}_L and \mathcal{U}_L are locally closed in $\mathbf{Gr}_{S_m}^q$.

(a) The intersection $W \cap \mathcal{U}_L$ is empty when $L \succ \mathbf{in}(W)$ and $W \cap \mathcal{U}_{\mathbf{in}(W)}$ is not empty by definition of initial extensor. Then, exploiting (i) we get that also $W \cap \mathcal{V}_L = \emptyset$ when $L \succ \mathbf{in}(W)$, and $W \cap \mathcal{V}_{\mathbf{in}(W)} = W \cap \mathcal{U}_{\mathbf{in}(W)}$ is a dense open subset of W .

To prove (b) we can apply the same arguments of the previous item and Proposition 3.1(i). \square

4. HILBERT SCHEME AND DOUBLE-GENERIC INITIAL IDEAL OF A GL-STABLE SUBSET

Let $p(t)$ be a Hilbert polynomial and denote by $\mathbf{Hilb}_{p(t)}^n$ the Hilbert scheme parameterizing the set of all subschemes with Hilbert polynomial $p(t)$ in the projective space \mathbb{P}_K^n . From now, we consider $\mathbf{Hilb}_{p(t)}^n$ as a subscheme of $\mathbf{Gr}_{S_m}^q$, where m is an integer larger than or equal to the Gotzmann number r of $p(t)$ and $q := \binom{n+m}{m} - p(m)$ (for instance, see [14]). Moreover, let \prec be a term order in S .

It is well-known that $\mathbf{Hilb}_{p(t)}^n$ is invariant under the action of GL, as a consequence of the definition of Hilbert scheme. Thus, for many aspects, we can consider the Hilbert scheme simply as a closed subscheme W of the Grassmannian, also stable under the action of GL, and can apply all the results we have obtained in Section 3 to its irreducible closed subsets that are stable under the action of GL. There is however an important issue that comes into play when $\mathbf{Hilb}_{p(t)}^n$ is involved. Roughly speaking, it is the relation between the notions of initial and generic initial extensors and the analogous ones for ideals. Now, we investigate this relation and show that, independently of the integer m , there is a well-defined ideal corresponding to the generic initial extensor of a closed irreducible subset of $\mathbf{Hilb}_{p(t)}^n$ that is also stable under the action of GL.

From now, a subset of $\mathbf{Hilb}_{p(t)}^n$ that is closed, irreducible, and stable under the action of GL is called a *GL-stable* subset.

Recall that every K -point of $\mathbf{Gr}_{S_m}^q$ is a q -dimensional K -vector space V of S_m . It is natural to consider the ideal generated by V in S and we denote it by I_V . Exploiting Theorem 2.7, we now relate the initial ideal $\mathbf{in}(I_V)$ and the generic initial ideal $\mathbf{gin}(I_V)$ to $\mathbf{in}(V)$ and $\mathbf{gin}(V)$, respectively.

In general, if V is any point of $\mathbf{Gr}_{S_m}^q$, $\mathbf{in}(I_V)$ does not need to coincide with the ideal $I_{\mathbf{vs}(\mathbf{in}(V))}$ and $\mathbf{gin}(I_V)$ does not need to coincide with $I_{\mathbf{vs}(\mathbf{gin}(V))}$, even though their homogeneous parts of degree m do, as shown by the following example.

Example 4.1. Let V be the vector space $\langle x_2^2, x_1x_2 + x_0^2 \rangle \subset K[x_0, x_1, x_2]_2$. For any term order in $K[x_0, x_1, x_2]$ with $x_0 \prec x_1 \prec x_2$, the initial extensor of V is $\mathbf{in}(V) = x_2^2 \wedge x_1x_2$ and the initial ideal of I_V is $\mathbf{in}(I_V) = (x_2^2, x_1x_2, x_0^2x_2, x_0^4)$. It is evident that x_2^2 and x_1x_2 do not generate $\mathbf{in}(I_V)$. Indeed, the Hilbert polynomial of $K[x_0, x_1, x_2]/(x_2^2, x_1x_2)$ is $t+2$, while the Hilbert polynomial of $K[x_0, x_1, x_2]/I_V$ and of $K[x_0, x_1, x_2]/\mathbf{in}(I_V)$ is $p'(t) = 4$.

If V is a point of a Hilbert scheme, an analogous situation to Example 4.1 cannot happen.

Theorem 4.2. *Let V be a K -point of $\mathbf{Hilb}_{p(t)}^n$, and set $V_1 := \mathbf{vs}(\mathbf{in}(V))$ and $V_2 := \mathbf{vs}(\mathbf{gin}(V))$. Then, the Hilbert polynomial of I_{V_1} and I_{V_2} is $p(t)$, and*

$$I_{V_1} = \mathbf{in}(I_V) = (\mathbf{in}(I_V)^{\text{sat}})_{\geq m}, \quad I_{V_2} = \mathbf{gin}(I_V) = (\mathbf{gin}(I_V)^{\text{sat}})_{\geq m}.$$

Proof. Recall that $\mathbf{Hilb}_{p(t)}^n$ is a closed subscheme of $\mathbf{Gr}_{S_m}^q$ and it is stable under the action of GL . Hence, we can apply Proposition 3.1(ii) to $\mathbf{Hilb}_{p(t)}^n$ and get that both V_1 and V_2 belong to $\mathbf{Hilb}_{p(t)}^n$. Therefore I_V , I_{V_1} and I_{V_2} share the same Hilbert polynomial $p(t)$.

Thinking of the points of $\mathbf{Hilb}_{p(t)}^n$ as subschemes of \mathbb{P}_K^n , the regularity of all of them is upper bounded by the Gotzmann number of $p(t)$, in particular by m . As a consequence, the saturated ideal in S defining a K -point of $\mathbf{Hilb}_{p(t)}^n$ can be completely recovered by saturation from its homogeneous part of degree m , then $(\mathbf{in}(I_V)^{\text{sat}})_{\geq m} = ((\mathbf{in}(I_V)^{\text{sat}})_m)$.

Thus, we obtain $I_{V_1} = \mathbf{in}(I_V)$ observing that these two ideals are generated in degree m , their Hilbert functions coincide in every degree $m' \geq m$, and $(I_{V_1})_m = \mathbf{in}(V)$ by Theorem 2.7. The equality $I_{V_2} = (\mathbf{gin}(I_V)^{\text{sat}})_{\geq m}$ follows by the same arguments. \square

Corollary 4.3. *Let W be a closed subset of $\mathbf{Hilb}_{p(t)}^n$. Then, the Hilbert polynomial of the ideals $I_{\mathbf{vs}(\mathbf{in}(W))}$ and $I_{\mathbf{vs}(\mathbf{gin}(W))}$ is $p(t)$. If, moreover, W is stable under the action of GL , then $I_{\mathbf{vs}(\mathbf{in}(W))} = I_{\mathbf{vs}(\mathbf{gin}(W))}$ is a point of W .*

Proof. By Proposition 2.9(ii) and possibly extending K to its algebraic closure as suggested in Remark 2.10, there is a K -point V of W such that $\mathbf{in}(V) = \mathbf{in}(W)$. Then we get the first statement for $\mathbf{in}(W)$ as a consequence of Theorem 4.2. This same argument applies to $\mathbf{gin}(W)$. The second statement directly follows by applying Proposition 3.1(i) and (ii) to the GL -stable subset $\overline{\mathcal{O}(\{V\})}$ of W . \square

A relevant and immediate consequence of the above results is that the points of the Hilbert scheme corresponding to the initial extensor and the generic initial extensor of anyone of its points do not depend on the Grassmannian in which we embed the Hilbert scheme, recalling that, for every integer $m \geq r$ where r is the Gotzmann number, the Hilbert scheme $\mathbf{Hilb}_{p(t)}^n$ can be embedded in $\mathbf{Gr}_{S_m}^q$. If we take $m' \geq r$, $m' \neq m$, we replace q by $q' := \binom{n+m'}{n} - p(m')$.

Corollary 4.4. *Let Z be a subset of $\mathbf{Hilb}_{p(t)}^n$ and denote by W and W' the images of Z by the embeddings of $\mathbf{Hilb}_{p(t)}^n$ in $\mathbf{Gr}_{S_m}^q$ and in $\mathbf{Gr}_{S_{m'}}^{q'}$, respectively, for some $m, m' \geq r$. Then,*

$$(I_{\mathbf{vs}(\mathbf{in}(W))})^{\text{sat}} = (I_{\mathbf{vs}(\mathbf{in}(W'))})^{\text{sat}} \quad \text{and} \quad (I_{\mathbf{vs}(\mathbf{gin}(W))})^{\text{sat}} = (I_{\mathbf{vs}(\mathbf{gin}(W'))})^{\text{sat}}.$$

Due to Corollary 4.4, we can finally give the following definition.

Definition 4.5. Let Y be a GL -stable subset of $\mathbf{Hilb}_{p(t)}^n$. We will denote by \mathbf{G}_Y the ideal $(I_{\mathbf{vs}(\mathbf{gin}(Y))})^{\text{sat}}$ in S and call it the *double-generic initial ideal* of Y .

Example 4.6. Given a Hilbert polynomial $p(t)$ with Gotzmann number r , an integer n and a term order \prec on S , let I be the ideal generated in S by the greatest $\binom{n+r}{n} - p(r)$ terms of degree r w.r.t. \prec . If $p(t)$ is the Hilbert polynomial of I , then I is a hilb-segment ideal (e.g. [15, Definition 3.7]). If the hilb-segment ideal exists (e.g. [15, Proposition 3.17]), it is the double-generic initial ideal \mathbf{G}_Y of every irreducible component Y of $\mathbf{Hilb}_{p(t)}^n$ containing it.

We end this section observing that, in addition to the irreducible components, there are many other relevant subsets of the Hilbert scheme that are invariant under the action of GL. We now list a few examples which are obtained applying Proposition 3.1.

Example 4.7.

- (i) The irreducible components of the singular locus of $\mathbf{Hilb}_{p(t)}^n$ are GL-stable.
- (ii) Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be the Hilbert function of a subscheme of \mathbb{P}_K^n and let W the locus of $\mathbf{Hilb}_{p(t)}^n$ of points corresponding to subschemes Z of \mathbb{P}_K^n whose Hilbert function H_Z behaves so that $H_V(t) \leq f(t)$ for every $t \in \mathbb{N}$. The irreducible components of W are GL-stable by Proposition 3.1. Indeed, W is stable under the action of GL and is closed by semicontinuity (for example, see [32]).
- (iii) For any given integer s , let W be the locus of $\mathbf{Hilb}_{p(t)}^n$ of points corresponding to subschemes of \mathbb{P}_K^n whose regularity is lower than or equal to s , that is the Hilbert scheme with bounded regularity $\mathbf{Hilb}_{p(t)}^{n,[s]}$ which is studied in [2]. It is obviously stable under the action of GL, and it is open by semicontinuity. Then, the irreducible components of the complementary $\mathbf{Hilb}_{p(t)}^n \setminus W$ (i.e. the set of points of $\mathbf{Hilb}_{p(t)}^n$ corresponding to subschemes with regularity $\geq s + 1$) are GL-stable.

Example 4.8. Let $p(t) = c$ be a constant Hilbert polynomial.

- (i) The locus of \mathbf{Hilb}_c^n of points corresponding to schemes that are supported on a unique point is closed and stable under the action of GL. Thus, its irreducible components are GL-stable.
- (ii) For $p(t) = c$, the locus of \mathbf{Hilb}_c^n of points corresponding to Gorenstein schemes is an open subset, stable under the action of GL. Thus, the irreducible components of its closure are GL-stable.
- (iii) For $p(t) = c$, the irreducible components of the closure of the locus of \mathbf{Hilb}_c^n of the Gorenstein schemes that are supported on a unique point are GL-stable.
- (iv) The irreducible components of the locus of \mathbf{Hilb}_c^n of points corresponding to non-reduced subschemes of \mathbb{P}_K^n are GL-stable.

Many other examples can be obtained considering every locus of $\mathbf{Hilb}_{p(t)}^n$ that is defined as a subscheme of \mathbb{P}_K^n by any property of its points, because such a locus is invariant under the action of GL.

5. THE PARTIAL ORDER \ll ON THE TERMS OF $\wedge^q S_m$

In this section, we introduce a partial order \ll between finite subsets with the same cardinality q of a totally ordered set T and prove some properties. Then, we will apply these results to the case of lists of terms in S_m and extend them to terms in $\wedge^q S_m$. We obtain that the double-generic initial ideal of a GL-stable subset Y of a Hilbert scheme is the maximum among all the Borel terms in Y with respect to the partial order \ll . This characterization gives a simple and deterministic method to recognize the double-generic initial ideal of Y among all the Borel ideals in Y .

Definition 5.1. Let (T, \prec) be a totally ordered set and consider $A, B \subseteq T$ containing q distinct elements each. We write $A \ll B$ if there is a bijection $\omega : A \rightarrow B$ such that $a \preceq \omega(a)$, for every $a \in A$.

It is quite obvious that \ll is a partial order and that in particular $A \ll A$. The following technical result allows a better understanding of its meaning.

Proposition 5.2. *Let (T, \prec) be a finite, ordered set and A, B be two subsets of T containing q distinct elements each. Further, we index the elements of $A = \{a_1, \dots, a_q\}$ so that $a_i \succeq a_{i+1}$ for every $i \in \{1, \dots, q-1\}$, and similarly for B . The followings are equivalent:*

- (i) $A \ll B$;
- (ii) for every element $c \in T$, $|\{a_i : a_i \succeq c\}| \leq |\{b_j : b_j \succeq c\}|$;
- (iii) for every $i \in \{1, \dots, q\}$, $a_i \preceq b_i$;
- (iv) $\{a_1, \dots, a_q\} \setminus \{b_1, \dots, b_q\} \ll \{b_1, \dots, b_q\} \setminus \{a_1, \dots, a_q\}$.

Proof. We first prove that item (i) implies item (ii). Observe that we can assume $c \in A \cup B$, by replacing c if necessary by the smallest term w.r.t. \prec in $A \cup B$ which is greater than c .

If $c = a_s \in A$, then there are exactly s elements in A bigger than or equal to c : more precisely, $a_i \preceq c$ for $i \in \{1, \dots, s\}$. Consider the bijection $\omega : A \rightarrow B$ such that $\omega(a_i) \succeq a_i$, for every $i \in \{1, \dots, q\}$. Since we have $\{b_i \in B | b_i \succeq a_s = c\} \supseteq \{\omega(a_j)\}_{j \in \{1, \dots, s\}}$ then it is immediate that $|\{b_i \in B | b_i \succeq a_s = c\}| \geq s$.

Otherwise, if $c = b_s \in B$, then $|\{b_j \in B | b_j \succeq b_s = c\}| = s$, and for every $j > s$ we have $b_s \succ b_j \succeq \omega^{-1}(b_j)$. Hence $|\{a_j \in A | a_j \succeq b_s = c\}| \leq s$ since it is contained in $A \setminus \{\omega^{-1}(b_{s+1}), \dots, \omega^{-1}(b_q)\}$.

Item (ii) implies (iii), by contradiction: if there is $j \in \{1, \dots, q\}$ such that $a_j \succ b_j$, then $|\{a_i \in A | a_i \succeq a_j\}| = j > |\{b_i \in B | b_i \succeq a_j\}|$.

Finally, if item (iii) holds, then we can consider the bijection $\omega : A \rightarrow B$ defined as $\omega(a_i) = b_i$, which fulfills the Definition of \ll .

The equivalence between item (ii) and (iv) is immediate. \square

Proposition 5.2(iii) points out a “natural” bijection between the sets A and B which fulfills Definition 5.1, but it does not mean that there are no other such bijections. If, for instance, $b_1 \succeq \dots \succeq b_q \succ a_1 \succeq \dots \succeq a_q$, then every bijection from A to B fulfills Definition 5.1.

If \prec is a term order on S , then for every integer m the couple $(S_m \cap \mathbb{T}, \prec)$ is a finite ordered set. From now, we identify every normal expression $\tau_1 \wedge \dots \wedge \tau_q \in \wedge^q S_m$ with the set $\{\tau_1, \dots, \tau_q\} \subset S_m \cap \mathbb{T}$.

Definition 5.3. For every two terms $\tau_1 \wedge \dots \wedge \tau_q$ and $\sigma_1 \wedge \dots \wedge \sigma_q \in \mathbf{T}_{S_m}^q$, we write $\tau_1 \wedge \dots \wedge \tau_q \ll \sigma_1 \wedge \dots \wedge \sigma_q$ if and only if $\{\tau_1, \dots, \tau_q\} \ll \{\sigma_1, \dots, \sigma_q\}$, according to Definition 5.1.

Now, we can apply the partial order \ll of Definition 5.3 and the results of Proposition 5.2 to the terms of $\wedge^q S_m$. If N is a set of terms of $\wedge^q S_m$, by $\max_{\ll} N$ we denote (if it exists) the maximum of N w.r.t. the order \ll .

In Remark 2.5, we observed that, for every point V of $\mathbf{Gr}_{S_m}^q$, we have $\mathbf{in}(V) \preceq \mathbf{gin}(V)$. Now, we observe there is a stronger relation between $\mathbf{in}(V)$ and $\mathbf{gin}(V)$. More generally, we prove that we can replace the order of (2.1) by the order \ll of Definition 5.3 in the study of initial and generic initial extensors of irreducible closed subsets of $\mathbf{Gr}_{S_m}^q$.

Theorem 5.4. *Let W be a subset of $\mathbf{Gr}_{S_m}^q$ such that \overline{W} is irreducible. Then*

- (i) $\mathbf{in}(W) = \max_{\ll} \Delta \text{Supp}(W)$;
- (ii) $\mathbf{gin}(W) = \max_{\ll} \Delta \text{Supp}(\mathcal{O}(W))$;
- (iii) $\mathbf{in}(W) \ll \mathbf{gin}(W)$;
- (iv) $\mathbf{gin}(W) = \max_{\ll} (\mathbf{B}_{S_m}^q \cap \overline{W})$.

Proof. For what concerns statement (i), thanks to Proposition 2.3(i) and Proposition 2.9(i), we can assume that W is closed. Furthermore, if V is the generic point of W , then $\mathbf{in}(V) = \mathbf{in}(W)$ and $\Delta\text{Supp}(V) = \Delta\text{Supp}(W)$ by Propositions 2.3(iii) and 2.9(iii). Up to an extension of the field of scalars, it is sufficient to prove statement (i) for the K -point V .

Let $\mathbf{in}(V) = \tau_1 \wedge \cdots \wedge \tau_q$. We show that $\sigma_1 \wedge \cdots \wedge \sigma_q \prec\prec \tau_1 \wedge \cdots \wedge \tau_q$, for every $\sigma_1 \wedge \cdots \wedge \sigma_q \in \Delta\text{Supp}(V) \setminus \{\mathbf{in}(V)\}$.

We can choose for the K -vector space V the unique basis $f_1, \dots, f_q \in S_m$, with $\mathbf{in}(f_i) = \tau_i$ and $f_i = \tau_i + \sum_{\eta_{j_i} \prec \tau_i} c_{ij_i} \eta_{j_i} = \sum_{\eta_{j_i} \preceq \tau_i} c_{ij_i} \eta_{j_i}$, where the coefficient of τ_i in the last summation is 1. Consider $f_1 \wedge \cdots \wedge f_q = \sum_{\sigma_1 \wedge \cdots \wedge \sigma_q \in \Delta\text{Supp}(V)} \Delta_L(V) \sigma_1 \wedge \cdots \wedge \sigma_q$.

For every normal expression $\sigma_1 \wedge \cdots \wedge \sigma_q \in \Delta\text{Supp}(V) \setminus \{\mathbf{in}(V)\}$, by construction of $f_1 \wedge \cdots \wedge f_q$ we have

$$\sigma_1 \wedge \cdots \wedge \sigma_q = \text{sgn}(\gamma) \eta_{j_{\gamma(1)}} \wedge \cdots \wedge \eta_{j_{\gamma(q)}},$$

for some γ permutation of $\{1, \dots, q\}$ and $\eta_{j_{\gamma(\ell)}}$ appearing with non-null coefficient in $f_{\gamma(\ell)}$.

Hence, we can consider the bijection $\omega : \{\sigma_1, \dots, \sigma_q\} \rightarrow \{\tau_1, \dots, \tau_q\}$ such that $\omega(\sigma_\ell) = \tau_{\gamma(\ell)}$. This bijection ω fulfills Definition 5.1.

To prove (ii), we observe that

$$\mathbf{gin}(W) = \mathbf{in}(\mathcal{O}(W)) = \mathbf{in}(\overline{\mathcal{O}(W)}) = \max_{\prec\prec} \Delta\text{Supp}(\overline{\mathcal{O}(W)}),$$

by definition of generic initial extensor, by Proposition 2.9 and by fact (i). Then, we conclude because $\Delta\text{Supp}(\overline{\mathcal{O}(W)}) = \Delta\text{Supp}(\mathcal{O}(W))$ by Proposition 2.3.

To prove (iii), now it is enough to observe that $\Delta\text{Supp}(W)$ is included in $\Delta\text{Supp}(\mathcal{O}(W))$.

Fact (iv) follows by the previous items and by Theorem 2.7(iii). \square

Remark 5.5. It is important to observe that *the statement of Theorem 5.4 does not hold true in the weaker hypothesis that the subset W is only closed*. Indeed, the generic initial extensors of its irreducible components do not need to be comparable by the partial order $\prec\prec$. This is a crucial point for the application of this result in subsection 6.1.

Let I and J be two saturated monomial ideals in S and assume that their Hilbert functions are equal in degree m , that is $\dim_K(I_m) = \dim_K(J_m) = q$. If $I_m = \langle \sigma_1, \dots, \sigma_q \rangle$ and $J_m = \langle \tau_1, \dots, \tau_q \rangle$, then we can compare the sets of terms $\{\sigma_1, \dots, \sigma_q\}, \{\tau_1, \dots, \tau_q\}$ w.r.t. $\prec\prec$, and if $\{\sigma_1, \dots, \sigma_q\} \prec\prec \{\tau_1, \dots, \tau_q\}$, by abuse of notation we will simply write $I_m \prec\prec J_m$.

This relation between the degree m components of the two ideals I and J cannot be considered as a relation between the two ideals. Indeed, if $m' \neq m$ then the components of degree m' of I and J may have different dimensions, hence they are no more comparable using $\prec\prec$.

We now present some interesting cases, where the relation $\prec\prec$ among the degree m parts of two monomial ideals lying on the same Hilbert scheme is preserved when passing to another degree. The first one concerns the double-generic initial ideal of a GL-stable subset and follows from Theorem 5.4.

Corollary 5.6. *Let Y be any GL-stable subset of $\mathbf{Hilb}_{p(t)}^n$ and r be the Gotzmann number of the Hilbert polynomial $p(t)$. If J is the saturated ideal defining a point of Y , then for every $m \geq r$*

$$(5.1) \quad J_m \prec\prec (\mathbf{G}_Y)_m.$$

Proof. Taking the embedding of Y in $\mathbf{Gr}_{S_m}^q$, the point defined by J is the K -vector space J_m . Then, $\wedge^q J_m = \mathbf{in}(J_m)$ belongs to $\Delta\text{Supp}(Y)$. By Theorem 5.4, we have $\mathbf{gin}(Y) = \mathbf{in}(Y) = \max_{\prec} \Delta\text{Supp}(Y)$; hence in particular $\wedge^q J_m \prec \mathbf{gin}(Y)$. We can conclude because $\mathbf{gin}(J_m)$ is the homogeneous component of degree m of the double-generic initial ideal \mathbf{G}_Y of Y , by Corollary 4.4. \square

Remark 5.7. Let Y be GL-stable. If r_Y is the maximum among the regularities of the points of Y , (5.1) holds true also for every integer m' , $r_Y \leq m' < r$, by [2, Theorem 1.2].

Let I and J be any two saturated monomial ideals defining points of the same Hilbert scheme. Now, we show that $J_m \prec I_m$ is equivalent to $J_{m+1} \prec I_{m+1}$ if J and I are Borel-fixed and \prec is the degrevlex term order. Moreover, if $p(t)$ is a constant polynomial, this result holds true for every term order on S . For every term $\tau = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$, we set $\min(\tau) := \min\{i \in \{0, \dots, n\} : \alpha_i \neq 0\}$.

We recall that if the monomial ideal J is Borel-fixed, and $J = J_{\geq m}$, with $m \geq \text{reg}(J^{\text{sat}})$, then J is a stable ideal (see [38, 39] and the references therein for details about stable ideals and their properties). We extend [32, Definition 2.7] to such an ideal: we call *growth-vector* of J_m the vector $gv(J_m) := (v_0, \dots, v_n)$, with $v_i := |\{\tau \in J_m \cap \mathbb{T}_m : \min(\tau) = i\}|$.

Lemma 5.8. *Let J be any m -regular Borel-fixed ideal with Hilbert polynomial $p(t)$. Then the growth-vector of J_m depends only on $p(t)$ and m .*

Proof. Let $v = (v_0, \dots, v_n)$ be the growth-vector of J_m . By [20, Lemma 1.1], for every $t \geq m$, we have

$$(5.2) \quad q(t) = \dim S_t - p(t) = \sum_{i=0}^n v_i \binom{t-m+i}{i}.$$

Since J is Borel-fixed and m -regular, then $J_{\geq m}$ is stable and the term x_n^m belongs to J_m [39, Proposition 4.4], and we obtain that $v_n = 1$. By induction v_j is the product of $j!$ times the leading coefficient of $q(t) - \sum_{i=j+1}^n v_i \binom{t-m+i}{i}$. \square

From now on, let J and I be m -regular Borel-fixed ideals in S such that $\text{Proj}(S/J)$ and $\text{Proj}(S/I)$ share the same Hilbert polynomial $p(t)$.

Given a term $\tau = x_0^{\alpha_0} \cdots x_n^{\alpha_n} \in \mathbb{T}$, in the following we let $\partial_{x_i}(\tau) := \alpha_i$.

Lemma 5.9. *Let \prec be the degrevlex term order on S . Assume $J_m \prec I_m$, and let $\omega: J \cap \mathbb{T}_m \rightarrow I \cap \mathbb{T}_m$ be any function such that $\tau \preceq \omega(\tau)$ for every $\tau \in J_m \cap \mathbb{T}$. If $\ell := \min(\tau)$ then $\ell = \min(\omega(\tau))$ and $\partial_{x_\ell}(\tau) \geq \partial_{x_\ell}(\omega(\tau))$.*

Proof. Since we are using the degrevlex term order, then $\min(\tau) \leq \min(\omega(\tau))$ for every $\tau \in J_m \cap \mathbb{T}$. Hence, $\min(\tau) = \min(\omega(\tau))$, because the growth vector of J_m is the same as the one of I_m by Lemma 5.8. The second part of the statement follows directly from the definition of the degrevlex term order. \square

Proposition 5.10. *If \prec is the degrevlex term order on S , then $J_m \prec I_m$ if and only if $J_{m+1} \prec I_{m+1}$.*

Proof. First, assume that $J_m \prec I_m$ and let $\omega_m: J \cap \mathbb{T}_m \rightarrow I \cap \mathbb{T}_m$ be a bijective function such that $\tau \preceq \omega_m(\tau)$.

Every term in J_{m+1} is of kind τx_ℓ for a unique $\tau \in J_m$ and $\ell \leq \min(\tau)$, and the same is for I_{m+1} [20, Lemma 1.1]. For every $\tau x_\ell \in J \cap \mathbb{T}_{m+1}$, we define $\omega_{m+1}(\tau x_\ell) := \omega_m(\tau) x_\ell$.

Since \prec is a term order, it is immediate that $\omega_{m+1}(\tau x_l) \succ \tau x_l$. Then, by Lemma 5.9 we see that the function $\omega_{m+1}: J \cap \mathbb{T}_{m+1} \rightarrow I \cap \mathbb{T}_{m+1}$ is bijective.

Vice versa, assume that $J_{m+1} \ll I_{m+1}$ and let $\omega_{m+1}: J \cap \mathbb{T}_{m+1} \rightarrow I \cap \mathbb{T}_{m+1}$ be a bijective function such that $\tau x_\ell \preceq \omega_{m+1}(\tau x_\ell)$. We now construct $\omega_m: J \cap \mathbb{T}_m \rightarrow I \cap \mathbb{T}_m$.

We observe that there is a bijection between the terms in J_m and the subset J'_{m+1} of terms in J_{m+1} that are divisible by the square of their minimal variable; indeed we obtain such a bijection associating to $\tau \in J_m \cap \mathbb{T}$ the term $\tau \cdot x_{\min(\tau)} \in J_{m+1}$. The same happens for the subset $I'_{m+1} \subset I_{m+1}$ defined in the same way as J'_{m+1} .

By Lemma 5.9, we obtain $\omega_{m+1}^{-1}(I'_{m+1}) \subseteq J'_{m+1}$. On the other hand I'_{m+1} and J'_{m+1} have the same cardinality (that of $J \cap \mathbb{T}_m$ and $I \cap \mathbb{T}_m$). Hence, $\omega_{m+1}^{-1}(I'_{m+1}) = J'_{m+1}$ and we obtain the bijection ω_m by setting $\omega_m(\tau) = \omega_{m+1}(\tau \cdot x_\ell)/x_\ell$ where $\ell := \min(\tau)$. \square

Proposition 5.11. *If $p(t) = d$ is a constant Hilbert polynomial, then $J_m \ll I_m$ if and only if $J_{m+1} \ll I_{m+1}$, for every term order \prec on S .*

Proof. Being the Hilbert polynomial constant, J_t and I_t contain all the terms of degree t in the variables x_1, \dots, x_n , for every $t \geq m$. Hence, we can consider only the terms τ with $\min(\tau) = 0$ and conclude by Proposition 5.2(iv) and Lemma 5.8. \square

6. APPLICATIONS

We always consider $\mathbf{Hilb}_{p(t)}^n$ embedded in $\mathbf{Gr}_{S_m}^q$ for some $m \geq r$, and, for the sake of simplicity, we denote by $\mathbf{B}_{S_m}^q \cap \mathbf{Hilb}_{p(t)}^n$ the set of Borel-fixed extensor terms corresponding to points of $\mathbf{Hilb}_{p(t)}^n$. Recall that *all* the terms of $\mathbf{B}_{S_m}^q \cap \mathbf{Hilb}_{p(t)}^n$ can be obtained by the algorithms presented in [15, 30] in characteristic 0, and in [6] for every characteristic.

In this section, we show how the properties of the generic initial ideal and of the partial term order \ll in $\mathbf{B}_{S_m}^q \cap \mathbf{Hilb}_{p(t)}^n$ can be used to investigate the topological structure and the rationality of the irreducible components of a Hilbert scheme. The following first result singles out a condition that every Borel-fixed ideal defining a point of a GL-stable subset must satisfy.

Proposition 6.1. *Let Y be a GL-stable subset of $\mathbf{Hilb}_{p(t)}^n$.*

- (i) *If $L \in \mathbf{B}_{S_m}^q \cap \mathbf{Hilb}_{p(t)}^n$ and $\mathbf{vs}(L) \not\ll (\mathbf{G}_Y)_m$, then $\mathbf{vs}(L) \notin Y$.*
- (ii) *If V is a K -point of $\mathbf{Hilb}_{p(t)}^n$ and $\mathbf{vs}(\mathbf{gin}(V)) \not\ll (\mathbf{G}_Y)_m$, then $V \notin Y$.*

Proof. The statement is an immediate consequence of Theorem 5.4 and of Corollary 5.6. Indeed, by these results, the m -degree homogeneous part of the double-generic initial ideal of Y determines the maximal term among the Borel terms in Y and, hence, among all the terms in Y . \square

From Proposition 6.1, we have therefore that if $\mathbf{vs}(L)$ belongs to a GL-stable subset Y , where L is a term, then necessarily $\mathbf{vs}(L) \ll (\mathbf{G}_Y)_m$.

6.1. Detection of different components in a Hilbert scheme. In this subsection, we see that some interesting lower bounds for the number of irreducible components of a Hilbert scheme spring out from the properties of the double-generic initial ideal and of the partial term order \ll .

Proposition 6.2. *Let \prec be a term order in S and M_{\prec} be the number of the maximal terms in $\mathbf{B}_{S_m}^q \cap \mathbf{Hilb}_{p(t)}^n$ w.r.t. \prec . Then, there are at least M_{\prec} irreducible components in $\mathbf{Hilb}_{p(t)}^n$.*

Proof. The statement follows directly by Propositions 3.1(ii) and 6.1 and Theorem 5.4. \square

Assuming $n > 2$, if $J \subset S$ is a Borel-fixed ideal, we denote by $\text{sat}_{x_0, x_1}(J)$ the ideal generated by the evaluations at $(1, 1, x_2, \dots, x_n)$ of the terms generators of J and call it the x_0, x_1 -saturation of J (see [36]). Denote by Λ the term in $\mathbf{B}_{S_m}^q \cap \mathbf{Hilb}_{p(t)}^n$ corresponding to the unique saturated lex-segment ideal in S whose Hilbert polynomial is $p(t)$.

Corollary 6.3. *If $\text{char}(K) = 0$ and $\text{sat}_{x_0, x_1}(\text{vs}(L)) \neq \text{sat}_{x_0, x_1}(\text{vs}(\Lambda))$ for every maximal term $L \in \mathbf{B}_{S_m}^q \cap \mathbf{Hilb}_{p(t)}^n \setminus \{\Lambda\}$, then there are at least $M_{\prec} + 1$ irreducible components in $\mathbf{Hilb}_{p(t)}^n$.*

Proof. It is enough to apply Proposition 6.2 and [36, Theorem 6]. \square

Remark 6.4. If L is a maximal term in $\mathbf{B}_{S_m}^q \cap \mathbf{Hilb}_{p(t)}^n$, then the corresponding ideal $(\text{vs}(L))$ is strongly stable, also if the field K has positive characteristic. Indeed, let $L' = \sigma_1 \wedge \dots \wedge \sigma_q$ be a Borel-fixed extensor term whose corresponding ideal is not strongly stable. Then, over any field of characteristic zero, $\tau_1 \wedge \dots \wedge \tau_q := \mathbf{gin}(\langle \sigma_1, \dots, \sigma_q \rangle)$ is a Borel term corresponding to a strongly stable ideal with Hilbert polynomial $p(t)$ and $\tau_1 \wedge \dots \wedge \tau_q \succ \sigma_1 \wedge \dots \wedge \sigma_q$. Therefore, in $\mathbf{B}_{S_m}^q \cap \mathbf{Hilb}_{p(t)}^n$ there is at least a term which is \succ of L' . Following the terminology introduced in [12], the ideal (τ_1, \dots, τ_q) is the *zero-generic initial ideal* of $(\sigma_1, \dots, \sigma_q)$.

The bounds of Proposition 6.2 and of Corollary 6.3 are not meaningful in two cases. The first is when $p(t)$ is a constant Hilbert polynomial, because then all the Borel-fixed ideals are on the same component, hence for every term order on S we will find a unique maximal term w.r.t. \prec . The second case is when the given term order \leq on S is the deglex one, because there is the unique maximal term corresponding to the lex-segment ideal. Anyway, in general we can get useful information, although the lower bound depends on the term order given on S , as the following example shows.

Example 6.5. For $n = 3$ and $p(t) = 7t - 5$, the Gotzmann number is $r = 16$. We get the complete list of the 112 strongly stable ideals in \mathbf{Hilb}_{7t-5}^3 by [15, 30] and compare their intersections with S_{16} w.r.t. \prec for several term orders in S . As just observed, there is only one maximal element for the lexicographic term order. If we consider the term order on S given by the weight vector $[w_0 = 1, w_1 = 2, w_2 = 9, w_3 = 12]$ (and ties broken by lex) we obtain two maximal terms corresponding to the ideals with saturations $\mathbf{b}_1 := (x_3^3, x_3^2 x_2, x_3 x_2^2, x_3^2 x_1, x_2^5)$ and $\mathbf{b}_2 := (x_3^2, x_3 x_2^3, x_2^4)$, respectively. Computing the x_0, x_1 -saturation, we see that neither of them lies on the component containing the lex-segment ideal, because $(\text{vs}(\Lambda))^{\text{sat}} = (x_3, x_2^8, x_1^7 x_2^9)$. Thus, there are at least 3 irreducible components in $\mathbf{Hilb}_{p(t)}^n$ by Corollary 6.3.

If we choose the degrevlex term order, we find 4 maximal terms corresponding to the ideals with the following saturations: the ideals $\mathbf{b}_1, \mathbf{b}_2$ previously considered and

$$\begin{aligned} \mathbf{b}_3 &:= (x_3^3, x_3^2 x_2^2, x_3 x_2^3, x_2^2 x_2 x_1, x_3 x_2^2 x_1, x_3^2 x_1^2, x_3 x_2 x_1^2, x_3 x_1^3, x_2^7), \\ \mathbf{b}_4 &:= (x_3^3, x_3^2 x_2, x_3 x_2^2, x_3^2 x_1^2, x_3 x_2 x_1^2, x_2^6). \end{aligned}$$

We conclude there are at least 4 irreducible components in $\mathbf{Hilb}_{p(t)}^n$, by Proposition 6.2 since in this case the hypothesis of Corollary 6.3 does not hold.

6.2. Maximal Hilbert function in a GL-stable subset. In this subsection, if f and g are two numerical functions, we say that f is greater than g if $f(t) \geq g(t)$, for every $t \in \mathbb{N}$, and write $f \geq g$.

As we have already recalled in Section 5, if a monomial ideal J is Borel-fixed, and $J = (J_m)$, with $m \geq \text{reg}(J^{\text{sat}})$, then J is stable.

Theorem 6.6. *Let \prec be the degrevlex term order in S . If V and V' are two K -points of $\text{Hilb}_{p(t)}^n$ such that $J := I_V$ and $I := I_{V'}$ are Borel-fixed ideals, then*

$$V \ll V' \Rightarrow \dim_K(J^{\text{sat}})_t \geq \dim_K(I^{\text{sat}})_t, \quad \forall t \geq 0.$$

In particular, if Y is a GL-stable subset of $\text{Hilb}_{p(t)}^n$, the Hilbert function of $\text{Proj}(S/\mathbf{G}_Y)$ is the maximum among the Hilbert functions of $\text{Proj}(S/H)$, where H varies among the saturated ideals defining points of Y .

Proof. It is enough to prove $\dim_K(J^{\text{sat}})_t \geq \dim_K(I^{\text{sat}})_t$ for every $t < m$, because V and V' are points of the same Hilbert scheme and m is an upper bound for the regularities of both J and I .

For every $t < m$, $\dim_K(J^{\text{sat}})_t$ is the number of terms of J_m which are divisible by x_0^{m-t} , because J is Borel-fixed. Since $V' \succ V$, we can apply Proposition 5.2 (ii) to $c = x_0^t x_1^{m-t}$ and see that the number of terms in $\text{vs}(V)$ divisible by x_0^{m-t} is larger than or equal to those in V' . Hence, we obtain $\dim_K(J^{\text{sat}})_t \geq \dim_K(I^{\text{sat}})_t$.

The last statement follows from Theorem 5.4(ii) and the fact that, for every homogeneous polynomial ideal H , $\text{gin}_{\prec}(H^{\text{sat}}) = \text{gin}_{\prec}(H)^{\text{sat}}$, being \prec the degrevlex term order. \square

Remark 6.7. By Theorems 6.6 and 5.4(ii) we get another method to find different irreducible components of a Hilbert scheme that consists in detecting the maximal Hilbert functions of projective schemes with a given Hilbert polynomial. This method might be easier to use than the detection of the maximal Borel terms w.r.t. the partial order \ll . However, the detection of the maximal Hilbert functions gives a lower bound on the number of irreducible components which is far from being sharp: for instance, in Example 6.5 we find 4 maximal Borel-fixed terms but there are only 2 maximal Hilbert functions.

Remark 6.8. The existence of the maximum among the Hilbert functions on a GL-stable subset of $\text{Hilb}_{p(t)}^n$ can be proved by semicontinuity in the following way, although we observe that Theorem 6.6 gives a constructive answer. Let Y be a GL-stable subset of $\text{Hilb}_{p(t)}^n$, m be an upper bound on the Castelnuovo-Mumford regularity of points in Y . We define the following numerical function $f : \mathbb{N} \rightarrow \mathbb{N}$

$$f(t) = \begin{cases} \max\{H_{S/I}(t) \mid I \in Y\}, & \text{if } 1 \leq t \leq m \\ p(t), & \text{otherwise.} \end{cases}$$

For every $1 \leq s \leq m$, the subset $A_s = \{I \in Y \mid H_{S/I}(s) \geq f(s)\}$ of Y is open by semicontinuity [29, Remark 12.7.1 in chapter III]. Hence $\cap_{s=1}^m A_s$ is an open subset of Y and it is non-empty, because every A_s is non-empty by construction of f . Thus, there is an open subset of ideals $I \in Y$ having maximal Hilbert function f .

It would be nice to find a result analogous to that of Theorem 6.6 for the deglex term order. We state the following conjecture.

Conjecture 6.9. Let Y be a GL-stable subset of $\mathbf{Hilb}_{p(t)}^n$ and \preceq be the deglex term order. Then, the Hilbert function of $\text{Proj}(S/\mathbf{G}_Y)$ is the minimum among the Hilbert functions of $\text{Proj}(S/H)$, where H varies among the ideals defining a point of Y .

6.3. Rational components of a Hilbert scheme. The following results deal with the rationality of irreducible components in a Hilbert scheme. The main tools we use are some of the features of double-generic initial ideals together with arguments introduced in [22, 31, 37].

Theorem 6.10. *Let Y be an isolated irreducible component of $\mathbf{Hilb}_{p(t)}^n$. If $\mathbf{gin}(Y)$ corresponds to a smooth point in Y , then Y is rational.*

Proof. Let $L := \mathbf{gin}(Y)$ and recall that \mathcal{V}_L is the set of points of $\mathbf{Gr}_{S_m}^q$ having L as initial extensor (see (3.1)). By Proposition 3.1(i), L is equal to $\mathbf{in}(Y)$ and, by Proposition 3.3, $\mathcal{V}_L \cap Y$ is a dense open subset of Y .

By [37, Lemma 3.2] and [31], the set of ideals in S having a given monomial ideal J as initial ideal w.r.t. any given term order can be endowed with a structure of homogeneous scheme X w.r.t. a non-standard grading. The reduced scheme structure X^{red} and the isolated irreducible components of X turn out to be homogeneous too [22, Corollary 2.7]. Moreover, X is connected, because every isolated irreducible component contains J , and every isolated irreducible component of X that is smooth at J is isomorphic to an affine space [22, Corollary 3.3].

We now apply these results to the monomial ideal $J = (\mathbf{vs}(L))$. Note that it is enough to consider the reduced scheme structure X^{red} because we deal with the isolated irreducible components of X that are smooth at the point J . By definition of Hilbert scheme, X^{red} and $(\mathcal{V}_L \cap \mathbf{Hilb}_{p(t)}^n)^{\text{red}}$ are isomorphic. Moreover, by Proposition 3.3(a), the set $(\mathcal{V}_L \cap \mathbf{Hilb}_{p(t)}^n)^{\text{red}}$ contains the open subset $Y' := \mathcal{V}_L \cap Y$ of Y which is one of the irreducible components of $(\mathcal{V}_L \cap \mathbf{Hilb}_{p(t)}^n)^{\text{red}}$ and, hence, preserves a homogeneous scheme structure. Being $\mathbf{vs}(L)$ a smooth point of Y , it is also smooth on Y' . Thus, Y' is isomorphic to an affine space and Y is rational. \square

As an application of Theorem 6.10 we recover the well-known fact that the irreducible component of a Hilbert scheme containing the unique saturated lex-segment ideal is rational (e.g. [31]). We now present a new result in the following corollary, where we focus on the 2-codimensional Cohen-Macaulay points of a Hilbert scheme.

Corollary 6.11. *Let $p(t)$ be a Hilbert polynomial of degree $n - 2$. Every irreducible component Y of $\mathbf{Hilb}_{p(t)}^n$ containing a point V corresponding to an arithmetically Cohen-Macaulay subscheme is rational.*

Furthermore, if $L = \mathbf{gin}(V)$, then \mathcal{V}_L is isomorphic to an affine space and \mathcal{U}_L is the Cohen-Macaulay locus of Y .

Proof. Recall that the subset C of $\mathbf{Hilb}_{p(t)}^n$ formed by the points corresponding to arithmetically Cohen-Macaulay subschemes is an open subset containing V [26, Théorème (12.2.1)(vii)]. Moreover, V and every other point defining an arithmetically Cohen-Macaulay subscheme of codimension 2 correspond to smooth points in $\mathbf{Hilb}_{p(t)}^n$ (see [24] for $n = 2$ and [21, Theorem 2(i)] for $n \geq 3$). Hence, $C \cap Y$ is a smooth, non-empty open subset of Y . It is well-known that if we choose the degrevlex term order, then also $\mathbf{gin}(I_V)^{\text{sat}} = \mathbf{gin}(I_V^{\text{sat}})$ defines an arithmetically Cohen-Macaulay subscheme of \mathbb{P}_K^n .

By Theorem 4.2, the ideal $\text{gin}(I_V^{\text{sat}})$ is the double-generic initial ideal \mathbf{G}_Y of Y , and its homogeneous component of degree m is $\text{vs}(L)$ with $L = \mathbf{gin}(Y) = \mathbf{gin}(V)$. Thus, Theorem 6.10 allows us to conclude that \mathcal{V}_L is an affine space and we observe that, by definition, V belongs to \mathcal{U}_L . Note that the result we obtain holds for every $V \in C \cap Y$, hence $\mathcal{U}_L = C \cap Y$. \square

Example 6.12. In this example we apply Theorem 6.10 to the double-generic initial ideal \mathbf{G}_Y of an irreducible component Y of a Hilbert scheme, which is smooth for Y , but which is not smooth for the Hilbert scheme.

Let us consider the Hilbert scheme \mathbf{Hilb}_{3t+2}^3 over a field of null characteristic. There are 4 saturated Borel ideals corresponding to points on this Hilbert scheme

$$\begin{aligned} \mathbf{b}_1 &:= (x_3, x_2^4, x_1^2 x_2^3), & \mathbf{b}_2 &:= (x_3^2, x_2 x_3, x_1 x_3, x_2^4, x_1 x_2^3), \\ \mathbf{b}_3 &:= (x_3^2, x_2 x_3, x_2^3, x_1^2 x_3), & \mathbf{b}_4 &:= (x_3^2, x_2 x_3, x_2^3, x_1 x_2^2). \end{aligned}$$

The ideal \mathbf{b}_1 corresponds to the lex-segment point of \mathbf{Hilb}_{3t+2}^3 . It is well-known that such a point belongs to a unique component, that we denote by Y_1 , and this component is rational. By a direct computation, we find that the dimension of Y_1 is 18 and that its general point corresponds to the union of a plane cubic curve and two isolated points. By [36, Theorem 6], we see that also \mathbf{b}_2 and \mathbf{b}_3 define points of Y_1 , while \mathbf{b}_4 does not.

If we choose the degrevlex term order, we find that \mathbf{b}_4 is the maximum w.r.t. \ll of the Borel ideals. By a direct computation involving marked schemes (see [16, 7, 9]), using for instance either the Singular library [13, 17] or the algorithm described in [8], we obtain that \mathbf{b}_4 is contained in two irreducible components Y_2 and Y_3 . Therefore, the point corresponding to \mathbf{b}_4 is not smooth for the Hilbert scheme. However, it turns out to be smooth for both Y_2 and Y_3 , and by Theorem 6.10 we get that Y_2 and Y_3 are both rational.

To complete the description, by a direct computation we find that the dimension of Y_2 is 12 and its general point corresponds to the disjoint union of a conic and a line.

Here are the generators of one of the ideals we obtain after a random specialization of the 12 free parameters:

$$\begin{aligned} f_1 &= x_3^2 + 990x_0^2 - x_2^2 - 3x_1x_2 - x_1x_3 - 2x_1^2 + 67x_0x_3 - 23x_0x_2 - 68x_0x_1 \\ f_2 &= x_2x_3 + 484x_0^2 - x_2^2 - 2x_1x_3 + 4x_1^2 + 22x_0x_3 - 88x_0x_1 \\ f_3 &= x_2^3 - 6538x_0^3 + 3x_0x_2^2 + 4x_1^2x_3 - x_2x_1^2 + 30x_0x_1x_2 + 8x_0x_1x_3 + 286x_0x_1^2 + 6x_1^3 - 4x_0^2x_3 - \\ &\quad 711x_0^2x_2 - 386x_0^2x_1 \\ f_4 &= x_1x_2^2 + 1913x_0^3 - 3x_0x_2^2 + x_1^2x_3 - 3x_0x_1^2x_2 - 3x_1x_2 + 2x_0x_1x_3 + 69x_0x_1^2 + 5x_1^3 - x_0^2x_3 + \\ &\quad 22x_0^2x_2 - 815x_0^2x_1 \end{aligned}$$

and the primary decomposition of this ideal

$$\begin{aligned} \mathbf{p}_1 &= (3x_1 + x_3 + 23x_0, x_2 - 2x_1 + 22x_0) \\ \mathbf{p}_2 &= (-2x_1 + x_3 + 22x_0 - x_2, 7x_1^2 - 2x_1x_2 + 72x_0x_1 - 7x_0x_2 + x_2^2 - 645x_0^2). \end{aligned}$$

The dimension of Y_3 is 15 and its general point corresponds to the union of a twisted cubic curve and a point. Here are the generators of one of the ideals we obtain after a random specialization of the 15 free parameters:

$$\begin{aligned} f'_1 &= x_3^2 + 37x_0^2 - 18x_1^2 - x_2^2 - 3x_1x_3 - 3x_1x_2 + 2x_0x_3 - 6x_0x_2 + 15x_0x_1 \\ f'_2 &= x_2x_3 + 31x_0^2 - 12x_1^2 - x_2^2 - 2x_1x_3 + 2x_1x_2 + x_0x_3 + 16x_0x_1 \\ f'_3 &= x_2^3 - 150x_0^3 + 86x_0x_1^2 - 24x_1^3 - 10x_1^2x_3 + 37x_0x_1x_3 + x_2x_1^2 - 7x_0x_1x_2 - 30x_0^2x_3 - x_0^2x_2 - 11x_0^2x_1 \\ f'_4 &= x_1x_2^2 - x_0^3 - 2x_0x_1^2 + x_0x_2^2 - 2x_1^2x_3 + 3x_0x_1x_3 - x_2x_1^2 - x_0x_1x_2 + 5x_0^2x_3 - 3x_0^2x_1 \end{aligned}$$

and the primary decomposition of this ideal

$$\mathbf{p}'_1 = (x_0 + x_1, x_2 - 8x_0, x_3 - 7x_0)$$

$$\mathbf{p}'_2 = (-x_1x_2 - 2x_1x_3 - 2x_0x_1 - x_0^2 + x_2^2 + 5x_0x_3, x_2x_3 + 30x_0^2 - 4x_1x_3 + x_1x_2 - 12x_1^2 + 6x_0x_3 + 14x_0x_1, x_3^2 + 36x_0^2 - 5x_1x_3 - 4x_1x_2 - 18x_1^2 + 7x_0x_3 - 6x_0x_2 + 13x_0x_1).$$

Finally, computing the marked schemes on \mathbf{b}_2 and \mathbf{b}_3 , we check that Y_1 , Y_2 and Y_3 are the only irreducible components of \mathbf{Hilb}_{3t+2}^3 .

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